# SUPPLEMENTARY INFORMATION

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# S1 Data

A field F (such as gravity or topography) may be expanded on a sphere as follows  $^{16}$ 

$$F(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left( C_{lm} \cos m\phi + S_{lm} \sin m\phi \right) P_{lm}(\cos \theta)$$
(S1)

where  $P_{lm}(\cos\theta)$  is a fully normalised associated Legendre function,  $\theta$  is colatitude,  $\phi$  is longitude and  $C_{lm}$  and  $S_{lm}$  are spherical harmonic coefficients of degree l and order m.

#### S1.1 Gravity Model

The description of Titan's gravitational field<sup>3</sup> is given as non-normalised, dimensionless potential coefficients  $C_{lm}^{g'}$ ,  $S_{lm}^{g'}$ . To convert these to fully normalised gravity coefficients,  $C_{lm}^{g}$ ,  $S_{lm}^{g}$ , we write

$$\{C,S\}_{lm}^{g} = \left((2-\delta_{0m})(2l+1)\frac{(l-m)!}{(l+m)!}\right)^{-\frac{1}{2}}(l+1)\frac{GM}{R^{2}}\{C,S\}_{lm}^{g'}$$
(S2)

where the square root term does the normalisation, the (l + 1) term arises from the differentiation associated with converting from potential to gravity, the  $\frac{GM}{R^2}$  term generates dimensional coefficients (which we will express in terms of mGal =  $10^{-5}$  ms<sup>-2</sup>), and  $\delta_{0m}$  is the Kronecker delta.

Three solutions have been derived<sup>3</sup> for Titan's gravity field. In the first two (SOL1a and SOL1b), results from six *Cassini* gravity flybys were analysed separately and then combined into multi-arc solutions. Whereas SOL1a attempts to model the gravity field only up to degree 3, SOL1b attempts to model the field up to degree 4, but only as a means of verifying the robustness of the degree-3 solution. It is found that the degree-3 solution differs only modestly between SOL1a and SOL1b. For the last solution (SOL2), a global model extending to l = 3 was derived using *Pioneer* and *Voyager* data and satellite ephemerides in addition to *Cassini* observations. In spite of the different approaches, the SOL2 field closely resembles the SOL1a field. The SOL1b solution matches less closely, likely because it also attempts to include the degree-4 component of the field. Although all three solutions give a consistent estimate of the periodic  $k_2$  Love number (with a central value of 0.6), the static part of the degree-4 field is currently not well constrained due to the limited nature of the observations. Our calculations will be based primarily on the SOL1a gravity field.

#### S1.2 Topography Model

The non-normalised topography coefficients  $C_{lm}^{h'}$ ,  $S_{lm}^{h'}$  are derived from a combination of radar altimetry and analysis of the overlapping regions of radar images, in a technique known as SAR topo<sup>9,13,14</sup>, and were used to derive shape

solutions up to l = m = 11. Several distinct solutions were produced, depending on where the harmonic expansion was truncated. We denote these solutions Deg4-exp, Deg5-exp, ... Deg11-exp, where the number indicates the highest degree and order used to fit the observations.

Due to the large gaps in *Cassini* radar coverage (Figure S1), topography models with power beyond degree 6 are not adequately constrained unless an *a priori* restriction is applied (minimise rms deviation from best-fit sphere). Even when *a priori* constraints are applied, the coefficients tend to be less stable when the model's expansion limit exceeds degree 6 (Figure S2). Likewise, the resulting admittance estimates tend to be most stable for the topography models with power limited to degree 6 or less (Figure S3).



Figure S1: *Cassini* radar-derived elevation data for Titan. Elevation is given relative to the 2575-km reference sphere.

For our purposes, we prefer to use the highest resolution data available without requiring *a priori* constraints in the model fits. We therefore primarily use the Deg6-exp model in our analysis. In the main text Fig. 1b and both parts of Fig. 3, we use the Deg6-exp topography model<sup>9</sup>, the coefficients for which are given in Table S1. To convert from non-normalised coefficients to fully normalised coefficients,  $C_{lm}^h$ ,  $S_{lm}^h$ , we write

$$\{C,S\}_{lm}^{h} = \left((2-\delta_{0m})(2l+1)\frac{(l-m)!}{(l+m)!}\right)^{-\frac{1}{2}}\{C,S\}_{lm}^{h'}$$
(S3)



Figure S2: Degree-3 normalised topography model<sup>9</sup> coefficients (with 1- $\sigma$  error bars) as a function of the maximum spherical harmonic degree allowed when fitting the data.



Figure S3: Degree-3 admittance estimates corresponding to the topography model coefficients shown in Figure S2 and the SOL1a gravity field<sup>3</sup>. Admittance estimates are obtained as described in section S3.1, using a Monte Carlo analysis (error bars illustrate the standard deviations of each Monte Carlo distribution).

Term Estimate		1-sigma error		
C00	2574750.0	6.0		
C10	0.0	5.8		
C11	23.1	5.2		
S11	18.5	5.8		
C20	-169.5	4.5		
C21	-17.8	5.4		
S21	31.8	7.0		
C22	120.8	4.6		
S22	20.1	4.6		
C30	-4.9	3.8		
C31	2.8	3.7		
S31	-21.3	5.6		
C32	-8.8	5.9		
S32	-2.9	5.9		
C33	0.0	7.2		
S33	14.3	7.2		
C40	-39.3	3.7		
C41	22.1	3.2		
S41	75.9	4.2		
C42	4.5	4.5		
S42	-26.8	4.5		
C43	16.7	0.0		
S43	-50.2	0.0		
C44	0.0	0.0		
S44	-47.3	0.0		
C50	28.6	3.3		
C51	-26.9	2.3		
S51	-14.0	4.7		
C52	37.1	6.2		
S52	-12.4	6.2		
C53	0.0	0.0		
S53	0.0	0.0		
C54	0.0	0.0		
S54	0.0	0.0		
C55	0.0	0.0		
S55	0.0	0.0		
C60	-2.5	3.1		
C61	8.9	2.5		
S61	-61.0	3.8		
C62	-24.1	0.0		
562	0.0	0.0		
063	0.0	0.0		
563	0.0	0.0		
064	0.0	0.0		
S64	0.0	0.0		
065	0.0	0.0		
565	0.0	0.0		
000	0.0	0.0		
566	0.0	0.0		

Table S1: Fully normalised Deg6-exp topography model  $^9$  coefficients (in metres).

# S2 Ice Shell Model

## S2.1 Origin of long-wavelength topography

Titan's shape is primarily determined by tides and rotation; the former resulting in elongation along the tidal axis, and the latter in an equatorial bulge (flattening at the poles). The degree-2 and degree-4 topography will also be affected by variations in ice shell thickness that arise due to tidal heating<sup>7</sup>. As discussed below, non-Newtonian flow within the lower part of the ice shell could cause degree-3 shell thickness variations to develop from a pattern that is initially confined to degrees 2 and 4. This may, in part, explain the source of the observed degree-3 topography. Since ongoing tidal heating will support the maintenance of shell thickness variations at degrees 2 and 4, those variations could persist even as lower crustal flow (discussed below) continues to generate shell thickness variations at degree 3.

Heterogeneities in the ice shell could also contribute to a departure from the purely degree-2 and -4 pattern predicted from tidal heating and could thus be responsible for part of the shell thickness variations, and therefore topography, at degree 3.

#### S2.2 Ice shell structure

We follow the common strategy of modelling the ice shell as an elastic layer overlying an inviscid layer. Roughly speaking, ice will undergo a transition from elastic to viscous behaviour at temperatures in the range 160 - 180 K, depending on the exact strain rate and grain size assumed <sup>30</sup>. For a conductive ice shell with basal temperature  $T_b = 270$  K, the elastic thickness (T) will then be 38-50% of the total shell thickness (d), while if  $T_b = 210$  K, then T will be 58-75% of d. While the transition from elastic to viscous behaviour will occur over some finite region, that region will be thin because of the very strong variation in viscosity with depth. Hence, our two-layer model is a good approximation.

Assuming a heat flux of  $F \approx 4 \,\mathrm{mW/m^2}$  through Titan's (conductive) ice shell<sup>7</sup>, and allowing thermal conductivity to vary with temperature<sup>31</sup>, we can estimate the shell's elastic thickness (*T*) according to  $T = 567 \ln (T_z/T_s)/F$ , where  $T_z$  is the temperature at which the shell transitions from elastic to viscous behaviour. The resulting estimated elastic thickness is  $T \approx 82 - 98 \,\mathrm{km}$ . Despite the highly approximate nature of this analysis, it yields an elastic thickness that is consistent with our estimates (see main text Fig. 3b).

Throughout our analysis, we assume the ice shell to be in an equilibrium state where the various forces (flexure within the elastic part of the shell, weight of the overlying topography, and buoyancy of the root) are in balance. This is reasonable because the vertical response time of the shell (analogous to postglacial rebound on Earth) should be fast compared with the loading timescale.

#### S2.3 Lateral flow in the ice shell

Shell thickness variations lead to flow in the lowermost, low-viscosity part of the shell, which will tend to smooth out any such variations. For a Newtonian fluid, the timescale  $(\tau)$  for removal of variations is given by<sup>26</sup>

$$\tau = \frac{\eta_b}{g\Delta\rho\delta^3k^2}\tag{S4}$$

where  $\eta_b$  is the viscosity at the base of the shell, g is the acceleration due to gravity,  $\Delta \rho$  is the density contrast between the shell and the fluid underneath,  $\delta$  is the effective channel thickness in which flow occurs and k is the wavenumber (k = l/R), where l is spherical harmonic degree and R is the planetary radius).

Assuming a linear temperature gradient and a thermal conductivity<sup>31</sup> which goes as c/T, where  $c = 567 \,\mathrm{W/m}$ , the effective channel thickness ( $\delta$ ) is given by<sup>26</sup>

$$\delta = \frac{R_g T_b d}{Q \ln \left(T_b / T_s\right)} \tag{S5}$$

where  $R_g$  is the gas constant, d is the shell thickness, Q is the activation energy and  $T_b$  and  $T_s$  are the basal and surface temperatures, respectively. Finally, the viscosity  $\eta_b$  is given by

$$\eta_b = \eta_{ref} \exp\left[\frac{Q}{R_g} \left(\frac{1}{T_b} - \frac{1}{T_{ref}}\right)\right]$$
(S6)

where the viscosity of ice is  $\eta_{ref}$  at a reference temperature  $T_{ref}$ . Table S2 gives  $\tau$  as a function of  $T_b$  for spherical harmonic degree 3. Here, we have assumed  $\eta_{ref} = 10^{14} \text{ Pas}$  at  $T_{ref} = 273 \text{ K}$ , Q = 60 kJ/mol,  $T_s = 90 \text{ K}$ , d = 100 km,  $g = 1.35 \text{ m/s}^2$ ,  $\Delta \rho = 80 \text{ kg/m}^3$  and R = 2575 km. For the range of  $T_b$  values explored,  $\delta = 3.4 \text{ km}$ . Rheological parameters<sup>32</sup> are subject to some uncertainty; nonetheless, the results of Table S2 serve to illustrate the main conclusion, which is that flow is slow if the ocean is sufficiently cold ( $T_b \leq 220 \text{ K}$ ). A temperature of 220 K corresponds to 25 wt% ammonia in a simple NH<sub>3</sub> – H<sub>2</sub>O system<sup>33</sup>.

Table S2: Viscosity and timescale for removal of degree-3 ice shell thickness variations as a function of temperature.

$T_b$ (K)	$\eta_b$ (Pa s)	$\tau$ (Myr)
273	$1.0 \times 10^{14}$	0.55
250	$1.1 \times 10^{15}$	6.34
230	$1.4 \times 10^{16}$	77.6
210	$2.8 \times 10^{17}$	1491

In practice, the rheology of ice may be non-Newtonian, in which case our flow timescales will be underestimates<sup>26</sup>, permitting larger values of  $T_b$ . An important consequence of non-Newtonian flow is that mode-coupling occurs: an

initially degree-2 or degree-4 pattern (e.g., due to tidal heating) will develop a degree-3 component as flow proceeds, thus potentially explaining the observed degree-3 signal.

Finally, we note that if tidal heating is indeed occurring, a balance may develop wherein shell thickness variations are being generated by tidal heating just as quickly as lateral flow is removing those variations. Such an equilibrium situation could be stable even if the relaxation timescales are relatively short, again permitting higher values of  $T_b$ .

# S3 Admittance

The admittance,  $Z_l$ , may be thought of as the ratio between gravity and topography at a particular wavelength<sup>34</sup>, and is typically measured in mGal/km. An equivalent method makes use of the coherence between the Bouguer gravity and topography<sup>35</sup>. However, as discussed below, the likelihood of gravity disturbances originating in the silicate core leads us to prefer the admittance technique in this case.

#### S3.1 Observed Admittance

We may define the cross-power spectrum  $D_{ij}(l)$  between two fields i and j as

$$D_{ij}(l) = \sum_{m=0}^{l} C^{i}_{lm} C^{j}_{lm} + S^{i}_{lm} S^{j}_{lm}$$
(S7)

With this definition, the estimated admittance, Z(l), and the correlation,  $\gamma(l)$ , between the gravity and surface topography, represented by subscripts g and h, are as follows<sup>15</sup>

$$Z(l) = \frac{D_{hg}(l)}{D_{hh}(l)}$$
(S8)

$$\gamma(l) = \frac{D_{hg}(l)}{\sqrt{D_{hh}(l)D_{gg}(l)}} \tag{S9}$$

If some fraction of the gravity signal is not correlated with the surface topography, then the coherence  $(\gamma^2)$  will be less than one. However, the crucial advantage of equation (S8) is that any such gravity noise does not affect the estimated admittance, Z. For the case of Titan, contributions to gravity from deeper interfaces (such as the silicate interior) are likely to be important, while contributions to the surface topography from these processes are likely negligible. An approach like that embodied in equation (S8), which is unaffected by noisy gravity, is essential for interpreting the limited observations available at Titan.

#### S3.1.1 Monte Carlo Analysis

The admittance estimates illustrated in the main text Fig. 2 were obtained through a Monte Carlo analysis. For each of the three gravity and three topography models (nine combinations in total), the correlations and estimated admittances were obtained from a distribution of 100,000 distinct admittance and correlation estimates, each of which was based on gravity and topography coefficients that were generated randomly according to the 1- $\sigma$  errors in the model coefficients. Individual correlations and admittance estimates were computed according to equations (S8) and (S9).

The admittance estimated based on the Monte Carlo analysis will have a slightly smaller magnitude than the admittance estimated directly from the coefficients (i.e., when uncertainties are ignored). This is because, as long as there is uncertainty in the topography coefficients, the mean of the distribution of  $D_{hh}$  (see equation S7) will always be greater than the value of  $D_{hh}$  obtained directly from the estimated topography coefficients (because if x is normally distributed, then  $E(x^2) > [E(x)]^2$ ). For example, if uncertainties are ignored, the admittance computed directly from the SOL1a gravity<sup>3</sup> coefficients and the Deg6-exp topography<sup>9</sup> coefficients is  $-39 \,\mathrm{mGal/km}$ , whereas when uncertainties are accounted for using a Monte Carlo analysis, the mean estimated admittance is  $-32 \,\mathrm{mGal/km}$ . We adopt the latter value because it is more conservative—more negative admittances would require higher magnitudes of erosion and/or larger elastic thicknesses.

## S3.2 Model Admittance

In order to interpret the observed admittance, we require a model for how the topography is supported. Here, we will assume that the topography is supported by some combination of shell thickness variations and flexure<sup>7</sup>.

#### Gravity Anomaly

The gravity anomaly at degree l due to a thin surface layer of amplitude  $h_l$  and density  $\rho_c$  is given by <sup>36</sup>

$$\Delta g_l^t = \frac{(l+1)}{(2l+1)} 4\pi G h_l \rho_c \tag{S10}$$

and similarly, the gravity anomaly due to a thin layer (a "root") of thickness  $r_l$  and density contrast  $\Delta \rho = \rho_m - \rho_c$  at the base of the shell, is given by

$$\Delta g_l^b = -\frac{(l+1)}{(2l+1)} 4\pi G r_l \Delta \rho \left(1 - \frac{d}{R}\right)^{l+2} \tag{S11}$$

where the mean thickness of the shell is d, the radius of the body is R, and  $\rho_m$  is the density of the material underlying the shell (i.e., the subsurface ocean). In the short-wavelength limit ( $l \gg 1$ ), equation (S10) reduces to the usual flat-plate formula, as required. When the net gravity anomaly and the surface topography  $(h_l)$  are known, the theoretical admittance is given by

$$Z_l = \frac{\Delta g_l^t + \Delta g_l^b}{h_l} \tag{S12}$$

In the remainder of this development, we drop the subscripts from both h and r and take it as understood that these parameters correspond to a specific wavelength.

In practice, it will be difficult to observe r and therefore to compute  $\Delta g_l^b$  according to equation (S11). Instead, we would like to find an expression for r in terms of h, which can be more readily observed. This is generally possible because, for a finite elastic thickness, there will be a pressure balance between the overburden of positive surface topography  $(\rho_c gh)$ , the buoyancy of the root  $(\Delta \rho gr)$ , and the restoring forces due to flexure.

#### Lithospheric Deflection, Cartesian case

If the deflection of the initial ice shell is w, in a Cartesian system, and if geoid undulations are neglected for the moment, this pressure balance can be written as

$$D\nabla^4 w = \Delta \rho gr - \rho_c gh \tag{S13}$$

Here, D represents flexural rigidity and is given by

$$D = \frac{ET^3}{12(1-\nu^2)}$$
(S14)

where T is the effective elastic thickness, E is Young's modulus and  $\nu$  is Poisson's ratio. We treat w as positive upward bending, h as positive upward relief above the reference ice shell surface, and r as positive downward relief from the base of the ice shell. The relationship between r and h depends on the elastic properties of the shell and the thickness of loads applied at the top  $(d_t)$ and bottom  $(d_b)$  of the shell (Figure S4);  $d_t$  is the thickness of material added at the surface (a negative value would indicate erosion), and  $d_b$  is the thickness of material added at the base of the ice shell (a positive value would indicate basal freezing, a negative value, basal melting). Our model represents the equilibrium state achieved after the lithosphere has finished deflecting in response to the applied load(s). The model also assumes that the ice shell properties do not change over time.

Our formulation is similar to previous work<sup>20,21</sup>, however, our sign convention differs slightly and, for simplicity, we assume that material added at the top or bottom of the shell is also of density  $\rho_c$ . Our formulation also differs in that we handle top and bottom loads simultaneously with w being the total deflection resulting from the combined effects of top and bottom loading.



Figure S4: Illustration of the influence of top loading (pale pink material,  $d_t$ ) and bottom loading (pale blue material,  $d_b$ ) on ice shell flexure (w), surface relief (h) and root thickness (r).

From Figure S4, we have

$$h = w + d_t \tag{S15}$$

$$r = d_b - w \tag{S16}$$

Then, assuming the loads are periodic and in-phase, we can solve (S13) for w, obtaining

$$w = \frac{\Delta \rho d_b - \rho_c d_t}{\rho_m + \mu} \tag{S17}$$

Here, we have introduced a parameter,  $\mu$ , which will serve as a shorthand for the flexural rigidity at a particular wavelength and gravity. In a Cartesian system,

$$\mu(k) = \frac{ET^3k^4}{12(1-\nu^2)g} \tag{S18}$$

where k is a wavenumber. The advantage of using this shorthand will become clear when we move from a Cartesian to a spherical system.

It is useful to define a compensation function,  $C(\rho)$ , that expresses the degree of compensation under flexural support compared with the case of pure isostasy. This can be defined as the ratio of the deflection, w, according to (S17) to the zero-rigidity deflection,  $w_0$ , obtained from (S17) when  $\mu = 0$ . That is,  $C = w/w_0$ , or

$$C(\rho) = \frac{1}{1 + \frac{\mu}{\rho}} \tag{S19}$$

When the elastic thickness, T, is zero, C = 1 (fully compensated). The parameter  $\rho$  is the density contrast that is resisting the flexure (i.e., related to

buoyancy, overburden pressure, or both). In the isostatic limit,  $\Delta \rho r = \rho_c h$ , whereas in the top loading case,  $\Delta \rho r = \rho_c h C(\Delta \rho)$ , while in the bottom loading case,  $C(\rho_c)\Delta\rho r = \rho_c h$ , as we will see. The theoretical value of C = 0 corresponds to the zero compensation case which occurs when the ice shell is infinitely rigid (i.e., as  $\mu \to \infty$ ). In this case, deflection (w) becomes zero (equation S17) and so, from (S15) and (S16),  $h = d_t$  and  $r = d_b$ . In this scenario, h and r are independent of one another and so both  $d_t$  and  $d_b$  must be specified in order to predict admittance. However, as long as C > 0, there will be some finite deflection and it will be possible to obtain r as a function of h.

If C > 0 and both h and  $d_t$  are specified, then from (S15), (S16), (S17) and (S19), it can be shown that

$$r = \frac{\rho_c h}{\Delta \rho} \left[ \frac{1 - \frac{d_t}{h}}{C(\rho_c)} + \frac{d_t}{h} \right]$$
(S20)

We have factored out  $\rho_c h/\Delta\rho$  in order to facilitate direct comparison with the isostatic case and because it will be convenient to do so when calculating admittance using (S10), (S11) and (S12). In the case where no material has been added to the surface (i.e.,  $d_t = 0$ , so that loading is purely from the bottom), this expression reduces to  $C(\rho_c)\Delta\rho r = \rho_c h$ . Similarly, it can be shown that if loading occurs purely from the top,  $\Delta\rho r = \rho_c h C(\Delta\rho)$ .

#### Lithospheric Deflection, spherical case

The foregoing gives correct values for r in the Cartesian case, which is appropriate for short wavelengths. However, in order to interpret admittance at very long wavelengths, we must consider the spherical case. Assuming the icy crust behaves as a thin elastic shell of radius R, equation (S13) becomes

$$D\nabla^{6}w + 4D\nabla^{4}w + ETR^{2}\nabla^{2}w + 2ETR^{2}w = R^{4}\left(\nabla^{2} + 1 - \nu\right)\left(\Delta\rho gr - \rho_{c}gh + \rho_{m}gh_{g}\right)$$
(S21)

where D is as in (S14). Here, we have adopted a modified version of previous approaches<sup>20,21</sup>, which themselves follow an earlier derivation<sup>19</sup>. The final term in (S21) accounts for the elevation or depression of the geoid ( $h_g$ , which we treat as positive upward) that occurs with loading of the ice shell. Here, we will adopt an approximation<sup>20</sup> to obtain  $h_g$ , namely, we assume that  $\left(1 - \frac{d}{R}\right)^{l+2} \approx 1$ , and that the geoid and gravitational acceleration do not change with depth in the shell (as has been pointed out<sup>37</sup>, this was an implicit assumption of the previous work<sup>20</sup>).

Having obtained  $h_g$ , and using (S15) and (S16), we rewrite (S21) as

$$\left[ \left( 1 - \frac{3\rho_m}{(2l+1)\overline{\rho}} \right)^{-1} \frac{ET}{R^2 g} \left( \frac{\frac{T^2(\nabla^6 + 4\nabla^4)}{R^2 12(1-\nu^2)} + \nabla^2 + 2}{\nabla^2 + 1 - \nu} \right) + \rho_m \right] w = \Delta \rho d_b - \rho_c d_t$$

where  $\overline{\rho}$  is the mean density of the body. If w is expressed in spherical harmonics, we can replace  $\nabla^2$  with -l(l+1) and solve for w, recovering equation (S17),

$$w = \frac{\Delta \rho d_b - \rho_c d_t}{\rho_m + \mu}$$

but with the flexural rigidity parameter now being

$$\mu(l) = \left(1 - \frac{3\rho_m}{(2l+1)\overline{\rho}}\right)^{-1} \frac{ET}{R^2 g} \left(\frac{\frac{T^2 \left[l^3 (l+1)^3 - 4l^2 (l+1)^2\right]}{R^2 12(1-\nu^2)} + l(l+1) - 2}{l(l+1) - (1-\nu)}\right)$$
(S22)

For a spherical system, it is also necessary to account for the ratio of surface areas at the top and bottom of the shell (since the buoyancy of the root depends on its volume, not its thickness). This effect complicates the derivations but it can be shown that, if  $d_t$  and h are specified, then the surface area correction leads to

$$r = \frac{\rho_c h}{\Delta \rho} \left[ \frac{1 - \frac{d_t}{h}}{C(\rho_c)} + \frac{d_t}{h} \right] \left( 1 - \frac{d}{R} \right)^{-2}$$
(S23)

As required, this expression reduces to the Cartesian equivalent as  $R \to \infty$ . This correction also partially relaxes the simplifying assumption we made to obtain the geoid height so that we now assume  $\left(1 - \frac{d}{R}\right)^l \approx 1$  rather than  $\left(1 - \frac{d}{R}\right)^{l+2} \approx 1$ .

Having obtained an expression for r as a function of h (which is possible as long as C > 0), we can now substitute (S23) into (S11) and combine with (S10) and (S12) to get an expression for admittance that depends on h and  $d_t$ , but not r

$$Z(l) = \frac{(l+1)}{(2l+1)} 4\pi G\rho_c \left[ 1 - \left(\frac{1 - \frac{d_t}{h}}{C(\rho_c)} + \frac{d_t}{h}\right) \left(1 - \frac{d}{R}\right)^l \right]$$
(S24)

Equation (S24) implicitly accounts for the root thickness (r), bottom load thickness  $(d_b)$ , and mantle density  $(\rho_m)$ , such that these terms do not appear in the final expression. Model admittance is sensitive to mantle density only insofar as the ratio  $\rho_m/\bar{\rho}$  influences the geoid, the effect of which is captured in  $C(\rho_c)$  via (S22) and (S19).

When C = 1 (pure isostasy), (S24) reduces to

$$Z(l) = \frac{(l+1)}{(2l+1)} 4\pi G\rho_c \left[ 1 - \left(1 - \frac{d}{R}\right)^l \right]$$
(S25)

and admittance will always be positive since  $(1 - d/R)^l$  must always be less than 1.



Figure S5: Admittance as a function of top load  $(d_t, \text{ where negative } d_t \text{ indicates} erosion)$  when h = 66 m, T = d = 200 km and assuming the properties listed in Table S3.

In general (i.e., when 0 < C < 1), admittance depends on topography (h), the amount of top and/or bottom loading  $(d_t \text{ and/or } d_b)$ , as well as the elastic thickness, T, and the mean shell thickness, d. For a given wavelength, mean shell thickness (d), elastic thickness (T), and a fixed, positive h, admittance is a positive linear function of  $d_t$  (Figure S5) and crosses zero when  $d_t = h \left(1 - \left(\frac{1}{C(\rho_c)} - 1\right)^{-1} \left(\left(1 - \frac{d}{R}\right)^{-l} - 1\right)\right)$ . When both top and bottom loading have taken place, and if h is known, admittance may be either positive or negative, and is uniquely defined if either  $d_t$  or  $d_b$  is specified.

It can be shown that, when loading occurs purely from the top (i.e.,  $d_b = 0$ ), admittance is independent of surface relief, h, and is necessarily positive. Based on (S24), when loading occurs purely from the bottom (i.e.,  $d_t = 0$ ), admittance is, again, independent of surface relief, h, and is positive as long as  $C(\rho_c) >$  $(1 - d/R)^l$ . For degree 3, this is always true for the parameters given in Table S3. Hence, degree-3 admittance may be negative only if  $d_t/h$  is negative (i.e., when erosion has occurred at topographic highs) or if substantially different parameter values are adopted. To obtain a negative degree-3 admittance with pure bottom loading would require an increase of ~ 30% in the ratio of Young's modulus to the ice shell density  $(E/\rho_c)$ . It is also possible to obtain a negative admittance without erosion for sufficiently large elastic thicknesses and sufficiently short wavelengths (e.g., for l = 6 when T > 350 km, or for l = 9 when T > 200 km).

Parameter	Symbol	Assumed Value
Poisson's ratio for ice	ν	0.25
Young's modulus for ice	E	9 GPa
Crustal (ice shell) density	$ ho_c$	$920\mathrm{kg/m^3}$
Mantle (subsurface ocean) density	$\rho_m$	$1000\mathrm{kg/m^3}$
Titan's mean density	$\overline{ ho}$	$1880\mathrm{kg/m^3}$
Titan's radius	R	$2575\mathrm{km}$
Acceleration due to gravity at the surface	g	$1.35 {\rm m/s^2}$

**T** 11 Go D 1 0

#### $\mathbf{S4}$ Results

#### S4.1 Degree-3 maps

Our results suggest that the negative admittance we observe at degree-3 is the result of negative gravity anomalies from large roots dominating over the positive gravity anomalies from the associated topography. We tested this scenario by computing, everywhere over the surface of Titan, the gravity anomaly implied by the observed topography and then comparing the result with the observations (Figure S6c). The gravity anomaly is obtained by multiplying the observed degree-3 topography (Figure S6a) by equation (S24), assuming  $T = d = 200 \,\mathrm{km}$  and a degree-3 erosion amplitude of 293 m (i.e., 293 m of erosion at the topographic peaks and 293 m of deposition in the valleys). This is the amount of erosion required to produce  $-39 \,\mathrm{mGal/km}$ , the admittance obtained directly (i.e., neglecting uncertainties) from the SOL1a gravity<sup>3</sup> and the Deg6exp topography<sup>9</sup> (see section S3.1.1). Figure S6b shows the resulting gravity anomaly, computed everywhere over the surface. The gravity field predicted through this procedure resembles the observed field (compare panels (b) and (c) in Figure S6). Assuming the mantle density given in Table S3, the implied root thickness amplitude is  $\sim 1.4$  km.

#### S4.2 Uncertainty in degree-3 erosion estimates

As illustrated in Figure S5, admittance is approximately a direct linear function of top load. Conversely, the top load required to produce a given admittance can be obtained by solving equation (S24) for  $d_t$ :

$$d_t = h\left(1 - \left(\frac{1}{C(\rho_c)} - 1\right)^{-1} \left[ \left(1 - \frac{Z(l)}{4\pi G\rho_c} \frac{(2l+1)}{(l+1)}\right) \left(1 - \frac{d}{R}\right)^{-l} - 1 \right] \right)$$
(S26)

Based on (S26), Figure S7 shows how the implied erosion (negative  $d_t$ ) varies with the estimated admittance given various combinations of shell thickness (d) and elastic thickness (T). The dashed black line corresponds to Z(3) =



Figure S6: Degree-3 topography and gravity maps: (a) Deg6-exp topography<sup>9</sup>; (b) gravity computed as described in section S4.1 (assuming T = d = 200 km and 293 m of erosion); (c) SOL1a gravity<sup>3</sup>.

-32 mGal/km, the admittance estimate obtained from the Monte Carlo analysis (i.e., accounting for uncertainties) based on the SOL1a gravity<sup>3</sup> field and the Deg6-exp topography<sup>9</sup> solution. This is also the admittance assumed in generating Fig. 3b in the main text. The  $\pm 11 \text{ mGal/km}$  uncertainty in that admittance estimate (see main text Fig. 2) translates to  $\pm 81 \text{ m}$  uncertainty in the erosion estimate when T = d = 200 km. Different combinations of T and d lead to slightly different uncertainties, but roughly  $\pm 30\%$  is typical.

Although, as we argued in section S2.2, the ice shell is not likely to be entirely elastic, adopting T = d leads to more conservative estimates of the magnitude of erosion. For example, the magnitude of erosion required to give rise to  $Z(3) = -32 \,\mathrm{mGal/km}$  is  $\sim 241 \,\mathrm{m}$  when we assume  $T = d = 200 \,\mathrm{km}$  and  $\sim 577 \,\mathrm{m}$  when we assume  $T = 100 \,\mathrm{km}$ ,  $d = 200 \,\mathrm{km}$ .



Figure S7: Magnitude of implied erosion as a function of admittance given several combinations of mean shell thickness (d) and elastic thickness (T). Negative top loads correspond to surface erosion. The dashed black line indicates the mean admittance estimate obtained from the Monte Carlo analysis based on the SOL1a gravity<sup>3</sup> and the Deg6-exp topography<sup>9</sup> (-32 mGal/km).

#### S4.3 Effect on tidal Love number, $k_2$

The tidal Love number,  $k_2$ , of Titan has been measured<sup>3</sup>, with a 2- $\sigma$  lower bound on  $k_2$  of 0.413-0.439. It is therefore important to check that the kind of rigid elastic lid that we are proposing does not contradict the observations. To do so, we constructed a highly simplified model for the interior of Titan (Table S4).

This is not meant to be realistic, but suffices to demonstrate our results. The density structure approximately satisfies the nominal moment of inertia

Layer	Outer	Rigidity	Viscosity	Density
	Radius	(GPa)	(Pa s)	$(kg/m^3)$
	(km)			
Solid Core	2110	3	$10^{21}$	2600
High-Pressure Ice	2275	3	$10^{21}$	1000
Ocean	2575 - d	0	0	1000
Rigid Outer Shell	2575	3	$10^{21}$	920

Table S4: Simple model of Titan's interior, used to determine the effect of a rigid shell on the tidal Love number,  $k_2$ .

constraint, while the low rigidity in the inner layers is designed to reproduce the observed  $k_2$ . We followed previous work<sup>38</sup> in calculating the model  $k_2$  values. For rigid shell thicknesses of d = 5, 100 and 200 km we obtained  $k_2$  values of 0.568, 0.519 and 0.413, respectively. Hence, adding a rigid shell of thickness 100 km or 200 km reduces  $k_2$  by 9% or 27%, respectively—not enough to contradict the observed  $k_2$  values<sup>3</sup>.

#### S4.4 Degree-2 admittance and fluid Love number

Degree-2 admittance analysis of Titan's ice shell is complicated by the fact that the body is tidally and rotationally distorted. Tidal/rotational distortion dominates the degree-2 gravity signal and also makes a large contribution to the degree-2 topography. If we assume that basal freezing, uplift and erosion processes act similarly at degrees 2 and 3, then ice shell thickness variations should be responsible for a portion of the observed degree-2 gravity and topography signals.

Separating the degree-2 gravity and topography signals into their hydrostatic (i.e., tidal/rotational) and non-hydrostatic (i.e., due to ice shell thickness variations) parts is necessarily an iterative process. We begin by estimating Titan's fluid Love number,  $h_{2f}$ , from the observed degree-2 gravity signal  $(J_2)$ according to

$$h_{2f} = 1 + \frac{6}{5} \frac{g}{R\omega^2} J_2 \tag{S27}$$

where R is Titan's mean radius, g is the mean surface gravity, and  $\omega$  is the angular frequency of rotation. Based on the observed gravity field<sup>3</sup>, we obtain  $h_{2f} \approx 2.0$ . This allows us to predict the expected hydrostatic topography,  $h_T$  according to

$$h_T = h_{2f} \frac{R^2 \omega^2}{g} \left[ \frac{1}{2} \left( 3\cos^2 \phi + 1 \right) \left( 1 - \cos^2 \theta \right) - \frac{5}{6} \right]$$
(S28)

which we then subtract from the observed topography to get the non-hydrostatic topography,  $h_{shell}$  (i.e., that which is due to variations in ice shell thickness). We then multiply  $h_{shell}$  by equation (S24) to estimate the gravity signal due to an ice

shell with anomalously deep roots (the large root size will be forced implicitly by our choices of C and  $d_t$ , both of which will be a function of mean shell thickness, d and elastic thickness, T). Having obtained this gravity anomaly due to ice shell thickness variations ( $\Delta g_{shell}$ ), we conclude that the portion of the gravity signal that is due to tidal distortion is  $\Delta g_{tidal} = \Delta g_{total} - \Delta g_{shell}$ . Finally, we use the newly obtained tidal gravity field to get an updated estimate for  $h_{2f}$ , again using (S27). After 3-4 iterations, our estimate for  $h_{2f}$  converges to the fourth decimal place, allowing us to separate, in a self-consistent way, the tidal/rotational and ice shell thickness contributions to the degree-2 gravity signal. The final estimate for  $h_{2f}$  depends on the assumed mean shell thickness (d) and elastic thickness (T), as illustrated in Figure S8. If we assume that T = 100 km and d = 200 km, then we obtain  $h_{2f} \approx 2.15$  (moment of inertia factor  $\sim 0.36$ ).



Figure S8: Estimate for Titan's fluid Love number,  $h_{2f}$ , as a function of mean shell thickness (d) and elastic thickness (T).

Using this fluid Love number, we can estimate the non-hydrostatic portions of the degree-2 topography and gravity signals. Figure S9 shows how the observed degree-2 gravity field (a) compares with the predicted field (b), where the predicted field is the sum of the estimated hydrostatic gravity (c), based on  $h_{2f} = 2.15$ , and the gravity anomaly expected from the estimated ice shell thickness variations (d), assuming an erosion amplitude of 577 m (obtained from the main text Fig. 3b assuming T = 100 km and d = 200 km). The amplitude of the estimated non-hydrostatic gravity is ~ 2 mGal while the estimated hydrostatic gravity amplitude is ~ 21 mGal. This is one measure of Titan's departure from hydrostatic equilibrium.



Figure S9: Degree-2 gravity maps centered on the sub-Saturnian point (180° longitude): (a) SOL1a gravity<sup>3</sup>; (b) total predicted gravity signal; (c) gravity signal caused by tidal/rotational distortion assuming  $h_{2f} = 2.15$ ; (d) gravity signal caused by ice shell thickness variations assuming T = 100 km and d = 200 km, and therefore 577 m of erosion.

## S4.5 Degree-4 predictions

Assuming once again that the observed topography is the result of uplift due to basal freezing and that the surface has experienced a similar magnitude of erosion at degrees 3 and 4, we can predict the admittance at degree 4. We first obtain the magnitude of degree-3 erosion  $(d_t)$  over a range of values for Tand d from the main text Fig. 3b. Using this same value of  $d_t$  for the degree-4 erosion amplitude, we then use equation (S24) to estimate the admittance over the same range of values for T and d. Figure S10 illustrates that the degree-4 admittance should be negative if the elastic thickness, T, accounts for most of the total shell thickness, d.

If, for example, T = d = 200 km, and we assume the same magnitude of erosion at degrees 3 and 4 (in this case, 241 m), then based on the observed topography<sup>9</sup> (Figure S11a), we obtain a degree-4 admittance of -5.1 mGal/km and we can compute the implied degree-4 gravity anomaly everywhere over the surface (Figure S11b). Although the amplitudes are similar, our result is spatially unlike the reported degree-4 gravity field<sup>3</sup> (Figure S11c). However, as noted in section S1.1 as well as in the main text, the degree-4 gravity field is not currently regarded as reliable; future gravity flybys are expected to improve the determination of the degree-4 field by a factor of two, providing a better test of our prediction. Note that if we assume instead that T = 100 km and d = 200 km (the corresponding erosion amplitude being 577 m), the greater compensation leads to a muted gravity signal and an admittance that approaches zero. Hence a weak degree-4 gravity signal, or a degree-4 admittance that is positive or only weakly negative, may be an indication that the elastic layer accounts for a smaller portion of the total shell thickness (Figure S10).



Figure S10: Degree-4 admittance predicted for a range of elastic thicknesses (T) and total shell thicknesses (d).



Figure S11: Degree-4 topography and gravity maps centered on the sub-Saturnian point (180° longitude): (a) Topography<sup>9</sup>; (b) Predicted gravity signal caused by ice shell thickness variations assuming T = d = 200 km and therefore 241 m of erosion; (c) SOL1b gravity<sup>3</sup> (not currently considered reliable at degree 4).

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